HL Paper 2 Mock A 2020 – WORKED SOLUTIONS v1

Section A

1. (a) 100 students

(b) $Q_1 = \frac{n}{4} = \frac{800}{4} = 200$, $Q_3 = \frac{3n}{4} = 3 \cdot \frac{800}{4} = 600$

$a = \text{mark}\left(Q_1\right) = \text{mark}\left(200\right) = 55$

$b = \text{mark}\left(Q_3\right) = \text{mark}\left(600\right) = 75$

Hence, $a = 55,\ b = 75$

2. (a) Value after 1 year $= 3000 \times 1.046$

Value after 2 years $= \left(3000 \times 1.046\right) \times 1.046 = 3000 \times 1.046^2$

Value after n years $= 3000 \times 1.046^n$

Thus, value after 7 years $= 3000 \times 1.046^7 = \$4110.01$

(b) $5000 = 3000 \times 1.046^x \Rightarrow 1.046^x = \frac{5}{3} \Rightarrow x \ln\left(1.046\right) = \ln\left(\frac{5}{3}\right)$

$$\Rightarrow x = \frac{\ln\left(\frac{5}{3}\right)}{\ln\left(1.046\right)} = 11.3584\ldots$$

The investment will exceed \$5000 after a minimum of 12 full years

Hence, $x = 12$

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3. (a) $\dfrac{x-4}{2x^2-x-1}=\dfrac{x-4}{(2x+1)(x-1)}=\dfrac{\mathrm{A}}{2x+1}+\dfrac{\mathrm{B}}{x-1},\ \mathrm{A},\ \mathrm{B}\in\mathbb{R}$

$$\frac{x-4}{(2x+1)(x-1)}=\frac{\mathrm{A}(x-1)}{(2x+1)(x-1)}+\frac{\mathrm{B}(2x+1)}{(2x+1)(x-1)}\ \Rightarrow\ x-4=\mathrm{A}(x-1)+\mathrm{B}(2x+1)$$

Let $x=1$:

$$1-4=\mathrm{A}(1-1)+\mathrm{B}(2(1)+1)\ \Rightarrow\ 3\mathrm{B}=-3\ \Rightarrow\ \mathrm{B}=-1$$

Let $x=0$:

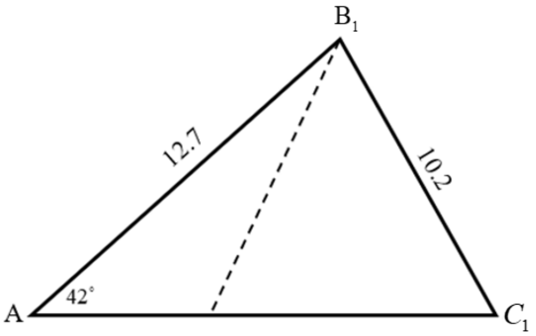
$$0-4=\mathrm{A}(0-1)-(2(0)+1)\ \Rightarrow\ -4=-\mathrm{A}-1\ \Rightarrow\ \mathrm{A}=3$$

Hence, $\dfrac{x-4}{2x^2-x-1}=\dfrac{3}{2x+1}-\dfrac{1}{x-1}$ Q.E.D.

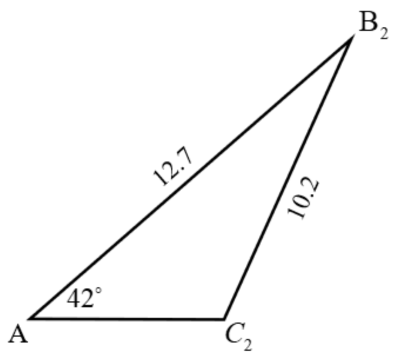
(b) $\displaystyle\int\frac{x-4}{2x^2-x-1}\,\mathrm{d}x=\int\left(\frac{3}{2x+1}-\frac{1}{x-1}\right)\mathrm{d}x=3\int\frac{1}{2x+1}\,\mathrm{d}x-\int\frac{1}{x-1}\,\mathrm{d}x$

$$\Rightarrow\ \int\frac{x-4}{2x^2-x-1}\,\mathrm{d}x=\frac{3}{2}\ln|2x+1|-\ln|x-1|+C$$

4.



OR



$\dfrac{\sin 42^\circ}{10.2}=\dfrac{\sin C_1}{12.7}\ \Rightarrow\ C_1=\sin^{-1}\left(\dfrac{12.7\sin 42^\circ}{10.2}\right)$

$C_1=56.442^\circ\ \Rightarrow\ B_1=180^\circ-(56.422^\circ+42^\circ)$

$B_1=81.578^\circ\ \Rightarrow\ \dfrac{\sin 81.578^\circ}{AC_1}=\dfrac{\sin 42^\circ}{10.2}$

$AC_1=\dfrac{10.2\sin 81.578^\circ}{\sin 42^\circ}=15.079\,\mathrm{cm}$

$C_2=180^\circ-56.422^\circ=123.578^\circ$

$B_2=180^\circ-(123.578^\circ+42^\circ)=14.422^\circ$

$\dfrac{\sin 14.222^\circ}{AC_2}=\dfrac{\sin 42^\circ}{10.2}$

$AC_2=\dfrac{10.2\sin 14.422^\circ}{\sin 42^\circ}=3.7966\,\mathrm{cm}$

Hence, the two possible lengths of AC are 15.1 cm and 3.80 cm

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5. (a) (i) binomial distribution: $n = 500,\ \ p = \frac{3}{5} = 0.6$

$$\mathrm{E}(X) = np = 500(0.6) = 300$$

(ii) standard deviation $= \sqrt{\mathrm{Var}(X)} = \sqrt{np(1-p)} = \sqrt{500(0.6)(0.4)} = 10.9545... \approx 11.0$

(b) $\mathrm{P}(300 - 10.9545... < X < 300 + 10.9545...) = \mathrm{P}(289.046... < X < 310.954...) \approx 0.662$

6. Let X be the random variable representing time (in minutes) it takes for a student to travel to school

$$\mathrm{P}(X < 5) = 0.04 \ \Rightarrow\ Z \approx -1.75069...$$

$$\mathrm{P}(X < 25) = 0.7 \ \Rightarrow\ Z \approx 0.524401...$$

Using formula for standardized normal variable $Z = \frac{x - \mu}{\sigma}$

$$-1.75069... = \frac{5 - \mu}{\sigma} \ \Rightarrow\ \mu - 1.75069\sigma = 5$$

$$0.524401... = \frac{25 - \mu}{\sigma} \ \Rightarrow\ \mu + 0.524401\sigma = 25$$

Solving system of linear equations: $\mu \approx 20.4\,\text{min},\ \ \sigma \approx 8.79\,\text{min}$

7. (a) Using GDC, $\frac{1+x}{(1-4x)^3}$ can be expressed as the sum of two fractions:

$$\frac{1+x}{(1-4x)^3} = \frac{5}{4(1-4x)^3} - \frac{1}{4(1-4x)^2} = \frac{5}{4}(1-4x)^{-3} - \frac{1}{4}(1-4x)^{-2}$$

By applying the binomial expansion theorem up to and including the x^3 term:

$$\frac{5}{4}(1-4x)^{-3} \approx \frac{5}{4}\left(1 + 12x + 96x^2 + 640x^3\right)$$

$$\frac{1}{4}(1-4x)^{-2} \approx \frac{1}{4}\left(1 + 8x + 48x^2 + 256x^3\right)$$

$$\Rightarrow\ \frac{5}{4}(1-4x)^{-3} - \frac{1}{4}(1-4x)^{-2} \approx \frac{5}{4}\left(1 + 12x + 96x^2 + 640x^3\right) - \frac{1}{4}\left(1 + 8x + 48x^2 + 256x^3\right)$$

$$= 1 + 13x + 108x^2 + 736x^3$$

Hence, $\frac{1+x}{(1-4x)^3} \approx 1 + 13x + 108x^2 + 736x^3$

(b) For the expansion to be valid, i.e. for it to converge:

$$|4x| < 1 \ \Rightarrow\ -1 < 4x < 1 \ \Rightarrow\ -\frac{1}{4} < x < \frac{1}{4}$$

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8. $v(t)=\int a(t)\,\mathrm{d}t=\int\left(\frac{3}{t}+2t\sin 2t\right)\mathrm{d}t=3\int\frac{1}{t}\mathrm{d}t+2\int t\sin 2t\,\mathrm{d}t$

To find $\int t\sin 2t\,\mathrm{d}t$, apply integration by parts:

$$\int u\,\mathrm{d}v=uv-\int v\,\mathrm{d}u$$

Let $u=t \Rightarrow \mathrm{d}u=\mathrm{d}t$, and let $\mathrm{d}v=\sin 2t\,\mathrm{d}t \Rightarrow v=\int\sin 2t\,\mathrm{d}t=-\frac{1}{2}\cos 2t$

$$\int t\sin 2t\,\mathrm{d}t=-\frac{t}{2}\cos 2t-\int -\frac{1}{2}\cos 2t\,\mathrm{d}t$$

$$=-\frac{t}{2}\cos 2t+\frac{1}{4}\sin 2t+C$$

Substitute back into our original equation, along with $3\int\frac{1}{t}\mathrm{d}t=3\ln t$:

$$\Rightarrow\ v(t)=3\ln t-t\cos 2t+\frac{1}{2}\sin 2t+C$$

At $t=1$, the particle is at rest, i.e. $v(1)=0$, so

$$v(1)=3\ln 1-\cos 2(1)+\frac{1}{2}\sin 2(1)+C=0$$

$$\Rightarrow\ C=\cos 2-\frac{1}{2}\sin 2=-0.8708\ldots$$

At $t=6$:

$$v(6)\approx 3\ln 6-6\cos 2(6)+\frac{1}{2}\sin 2(6)-0.8708=-0.8269\ldots$$

Hence, $v(6)\approx -0.827\,\mathrm{m\,s^{-1}}$

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9. $px^2+qx+q=0 \Rightarrow x^2+\frac{q}{p}x+\frac{q}{p}=0$

α and β are roots of the equation, so

$$x^2+\frac{q}{p}x+\frac{q}{p}=(x-\alpha)(x-\beta)=x^2-(\alpha+\beta)x+\alpha\beta$$

$$\Rightarrow \alpha+\beta=-\frac{q}{p},\ \alpha\beta=\frac{q}{p}$$

The equation with roots $\frac{1}{\alpha+1}$ and $\frac{1}{\beta+1}$ can be expressed as

$$\left(x-\frac{1}{\alpha+1}\right)\left(x-\frac{1}{\beta+1}\right)=0 \Rightarrow x^2-\left(\frac{1}{\alpha+1}+\frac{1}{\beta+1}\right)x+\left(\frac{1}{\alpha+1}\right)\left(\frac{1}{\beta+1}\right)=0$$

Focusing on the constant term:

$$\frac{1}{\alpha+1}\cdot\frac{1}{\beta+1}=\frac{1}{(\alpha+1)(\beta+1)}=\frac{1}{\alpha\beta+\alpha+\beta+1}=\frac{1}{\frac{q}{p}-\frac{q}{p}+1}=1$$

For the x term:

$$\frac{1}{\alpha+1}+\frac{1}{\beta+1}=\frac{\beta+1}{(\alpha+1)(\beta+1)}+\frac{\alpha+1}{(\alpha+1)(\beta+1)}=\alpha+\beta+2=-\frac{q}{p}+2$$

Rewriting the new equation:

$$x^2-\left(-\frac{q}{p}+2\right)x+1=0$$

Since we want integer coefficients, and $p,\ q\in\mathbb{Z}$, multiply through by p:

$$px^2+(q-2p)x+p=0 \qquad \text{Q.E.D.}$$

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Section B

10. (a) Input data into GDC to determine the linear regression equation L_1 :

$y = 10.7x + 121$ (values accurate to 3 significant figures)

(b) (i) gradient of regression equation is additional cost per box, i.e. **unit cost**

(ii) y-intercept of regression equation is the **fixed costs**, i.e. cost when zero boxes are produced

(c) $y = 10.6555(60) + 120.794 = 760.124$

Hence, cost of 60 boxes is approximately $760

(d) $19.99x > y = 10.6555x + 120.794$

$\Rightarrow 9.3345x > 120.794 \Rightarrow x > 12.9405\ldots$

Hence, the factory must produce at least 13 boxes per day to make a profit

(e) This would be extrapolation, which is not appropriate

(f) $\mathrm{L}_2 :\ x = 0.0844y - 7.88$

(g) $x = 0.08837(550) - 7.88307 = 40.72043$

Hence, approximately 41 boxes are produced when total production cost is $550

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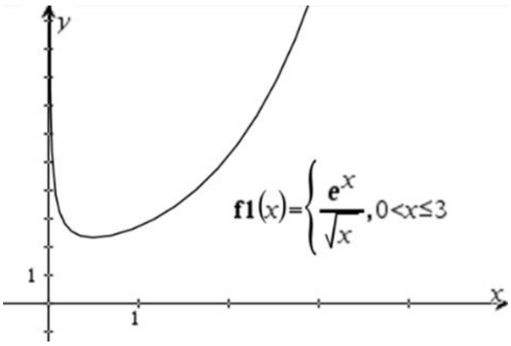
11. (a) (i)



(ii) $h(x)=\dfrac{e^x}{\sqrt{x}}=\dfrac{e^x}{x^{\frac{1}{2}}} \;\Rightarrow\; h^{-1}(x)=\dfrac{x^{\frac{1}{2}}e^x-\frac{1}{2}x^{-\frac{1}{2}}e^x}{\left(x^{\frac{1}{2}}\right)^2}=\dfrac{e^x\left(\sqrt{x}-\frac{1}{2\sqrt{x}}\right)}{x}=\dfrac{e^x\left(\frac{2x-1}{2\sqrt{x}}\right)}{x}=e^x\left(\dfrac{2x-1}{2x\sqrt{x}}\right)$

(iii) gradient of normal to curve is $-\dfrac{2x\sqrt{x}}{e^x(2x-1)}=\dfrac{2x\sqrt{x}}{e^x(1-2x)}$

(b) (i) gradient of (PQ) is $\dfrac{y-0}{x-1}=\dfrac{\frac{e^x}{\sqrt{x}}-0}{x-1}=\dfrac{e^x}{\sqrt{x}}\cdot\dfrac{1}{x-1}=\dfrac{e^x}{\sqrt{x}(x-1)}$

(ii) Equating the two expressions for gradient of normal to the curve gives

$\dfrac{e^x}{\sqrt{x}(x-1)}=\dfrac{2x\sqrt{x}}{e^x(1-2x)} \;\Rightarrow\; x\approx 0.545428...$ this is the x-coordinate of P

y-coordinate of P is $h(0.545428...)=\dfrac{e^x}{\sqrt{0.545428...}}\approx 2.33619...$

minimum distance from Q to graph of h is length of PQ

hence, minimum distance $=\sqrt{(0.545428...-1)^2+(2.33619...-0)^2}\approx 2.380001...$

minimum distance from Q to graph of h is approximately 2.38

(c) $g(x)=\dfrac{e^x}{c\sqrt{x}}=\dfrac{1}{c}\cdot h(x) \;\Rightarrow\; g'(x)=\dfrac{1}{c}\cdot h'(x)=\dfrac{e^x}{c}\left(\dfrac{2x-1}{2x\sqrt{x}}\right)$

The point on the graph of g nearest to point R on the x-axis is the point on g that has a horizontal tangent (parallel to x-axis); i.e. $g'(x)=0$. This point on g and point R have the same x-coordinate.

$g'(x)=\dfrac{e^x}{c}\left(\dfrac{2x-1}{2x\sqrt{x}}\right)=0 \;\Rightarrow\; 2x-1=0 \;\Rightarrow\; x=\dfrac{1}{2}$; Thus, point R is located at $\left(\dfrac{1}{2},0\right)$

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12. (a) k will always be positive. The temperature of the egg is decreasing which means that $\frac{\mathrm{dT}}{\mathrm{d}t} < 0$; and since $\mathrm{T} - \mathrm{A} > 0$ then it must follow that $-k < 0$; hence, $k > 0$

(b) (i) $\frac{\mathrm{dT}}{\mathrm{d}t} = -k(\mathrm{T}-18) \Rightarrow \frac{1}{\mathrm{T}-18}\mathrm{dT} = -k\,\mathrm{d}t \Rightarrow \int \frac{1}{\mathrm{T}-18}\mathrm{dT} = -k\int \mathrm{d}t$

$\ln(\mathrm{T}-18) = -kt + C_1 \Rightarrow \mathrm{e}^{\ln(\mathrm{T}-18)} = \mathrm{e}^{-kt+C_1} \Rightarrow \mathrm{T}-18 = \mathrm{e}^{-kt}\mathrm{e}^{C_1}$ let $C = \mathrm{e}^{C_1}$

$\mathrm{T}(t) = C\mathrm{e}^{-kt} + 18$; given $\mathrm{T}(0) = 98$ then $\mathrm{T}(0) = C\mathrm{e}^{0} + 18 = 98 \Rightarrow C = 80$

Thus, $\mathrm{T}(t) = 80\mathrm{e}^{-kt} + 18$ ***Q.E.D.***

(ii) Given $\mathrm{T}(5) = 38$, find k: $38 = 80\mathrm{e}^{-k(5)} + 18 \Rightarrow 80\mathrm{e}^{-5k} = 20 \Rightarrow \mathrm{e}^{-5k} = 0.25$

$\ln\left(\mathrm{e}^{-5k}\right) = \ln(0.25) \Rightarrow -5k = \ln(0.25) \Rightarrow k = \frac{\ln(0.25)}{-5} \approx 0.277259...$

$20 = 80\mathrm{e}^{-0.277t} + 18 \Rightarrow \mathrm{e}^{-0.277t} = 0.025 \Rightarrow \ln\left(\mathrm{e}^{-0.277t}\right) = \ln(0.025)$

$-0.277t = \ln(0.025) \Rightarrow t = \frac{\ln(0.025)}{-0.277259...} \approx 13.3048...$

Thus, it takes approximately 13.3 minutes for the egg to cool to $20\,^{\circ}\mathrm{C}$.

(c) (i) substituting, gives $\frac{\mathrm{dT}}{\mathrm{d}t} = -0.25\left(\mathrm{T} - 18\mathrm{e}^{-0.2t}\right)$; then $\frac{\mathrm{dT}}{\mathrm{d}t} = -0.25\mathrm{T} + 4.5\mathrm{e}^{-0.2t}$ ***Q.E.D.***

(ii) this is a first order linear differential equation: $\frac{\mathrm{dT}}{\mathrm{d}t} + 0.25\mathrm{T} = 4.5\mathrm{e}^{-0.2t}$

integrating factor is $\mathrm{e}^{\int 0.25\,\mathrm{d}t} = \mathrm{e}^{0.25t}$; multiply both sides of diff eqn by integrating factor – and applying product rule for differentiation 'backwards' on left side, gives

$\mathrm{e}^{0.25t}\left(\frac{\mathrm{dT}}{\mathrm{d}t} + 0.25\mathrm{T}\right) = \mathrm{e}^{0.25t}\left(4.5\mathrm{e}^{-0.2t}\right) \Rightarrow \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{0.25t}\mathrm{T}\right) = 4.5\mathrm{e}^{0.05t}$

integrate both sides w.r.t. t: $\int\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{0.25t}\mathrm{T}\right)\right]dt = 4.5\int \mathrm{e}^{0.05t}dt \Rightarrow \mathrm{e}^{0.25t}\mathrm{T} = 4.5\left(20\mathrm{e}^{0.05t}\right) + C$

$\frac{\mathrm{e}^{0.25t}\mathrm{T}}{\mathrm{e}^{0.25t}} = \frac{90\mathrm{e}^{0.05t} + C}{\mathrm{e}^{0.25t}} \Rightarrow \mathrm{T}(t) = 90\mathrm{e}^{-0.2t} + C\mathrm{e}^{-0.25t}$

$\mathrm{T}(0) = 98$: $98 = 90\mathrm{e}^{0} + C\mathrm{e}^{0} \Rightarrow C = 8$

Thus, $\mathrm{T}(t) = 90\mathrm{e}^{-0.2t} + 8\mathrm{e}^{-0.25t}$

(iii) $90\mathrm{e}^{-0.2t} + 8\mathrm{e}^{-0.25t} = 20 \Rightarrow t \approx 7.81242...$

Thus, it takes approximately 7.81 minutes for the egg to cool to $20\,^{\circ}\mathrm{C}$.